2023-24 MATH2048: Honours Linear Algebra II

Homework 5

Due: 2023-10-16 (Monday) 23:59

For the following homework questions, please give reasons in your solutions. Scan your solutions and submit it via the Blackboard system before due date.

1. For each of the following pairs of ordered bases β and β' for $P_2(R)$, find the change of coordinate matrix that changes β'-coordinates into β-coordinates.

   (a) $\beta = \{x^2, x, 1\}$ and $\beta' = \{a_2x^2 + a_1x + a_0, b_2x^2 + b_1x + b_0, c_2x^2 + c_1x + c_0\}$

   (b) $\beta = \{1, x, x^2\}$ and $\beta' = \{a_2x^2 + a_1x + a_0, b_2x^2 + b_1x + b_0, c_2x^2 + c_1x + c_0\}$

2. For each matrix A and ordered basis β, find $[L_A]_\beta$. Also, find an invertible matrix Q such that $[L_A]_\beta = Q^{-1}AQ$.

   (a) $A = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$

   (b) $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$

3. Let $T : V \to W$ be a linear transformation from an n-dimensional vector space V to an m-dimensional vector space W. Let β and γ be ordered bases for V and W, respectively. Prove that $\text{rank}(T) = \text{rank}(L_A)$ and that $\text{nullity}(T) = \text{nullity}(L_A)$, where $A = [T]_\beta^\gamma$. Hint: Apply Exercise 17 to Figure 2.2.

4. Let $c_0, c_1, \ldots, c_n$ be distinct scalars from an infinite field F. Define a transformation $T : P_n(F) \to F^{n+1}$ by $T(f) = (f(c_0), f(c_1), \ldots, f(c_n))$. Prove that T is an isomorphism. Hint: Use the Lagrange polynomials associated with $c_0, c_1, \ldots, c_n$.

5. Consider a linear transformation $T : V \to W$, where $\dim(V) = \dim(W) = n$. Show that if T has a left inverse U, then U is also a right inverse of T, thus T is invertible. (Hint. Sec. 2.4 Q10(b), prove it if you use it)

The following are extra recommended exercises not included in homework.

1. Summarize the concepts met after HW2. Make a mindmap of the concepts, definitions, examples and theorems. Make sure you remember the definitions.

2. Let $T : V \to Z$ be a linear transformation of a vector space V onto a vector space Z. Define the mapping $\bar{T} : V/N(T) \to Z$ by $\bar{T}(v + N(T)) = T(v)$ for any coset $v + N(T)$ in $V/N(T)$.

   (a) Prove that $\bar{T}$ is well-defined; that is, prove that if $v + N(T) = v' + N(T)$, then $T(v) = T(v')$.

   (b) Prove that $\bar{T}$ is linear.

   (c) Prove that $\bar{T}$ is an isomorphism.

   (d) Prove that the diagram shown in Figure 2.3 commutes; that is, prove that $\bar{T} = T\eta$. Here $\eta : V \to V/N(T)$ is the natural projection. (See the Figure 2.3 on the book at §2.4 Q24)

3. Consider a linear transformation $T : V \to W$. Prove or disprove the following.

   (a) If T has a right inverse, must it have a left inverse?

   (b) If T has a left inverse, must it have a right inverse?

   (c) If T has both a left and a right inverse, must it be invertible? (That is, must the left and right inverse be the same?)

   (d) If T has a unique right inverse S, is T necessarily invertible? (Hint. Consider $ST + S - I$.)

4. Let $g_0(x) = x + 1$. Let $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ and $U : P_3(\mathbb{R}) \to \mathbb{R}^3$ be defined by

$$T(f(x)) = f'(x)g_0(x) + \int_0^x f(t)dt \text{ and } U(h(x)) = (h(0), h(1), h'(1))$$

   Let α, β, γ be the standard ordered bases for $P_2(\mathbb{R}), P_3(\mathbb{R}), \mathbb{R}^3$ respectively.

   (a) Compute $[T]_\alpha^\beta$, $[U]_\beta^\gamma$, $[U]_\beta^\gamma[T]_\alpha^\beta$ and $[UT]_\alpha^\gamma$.

   (b) Let $h_0(x) = 1 - 2x - x^2 + x^3$, compute $[h_0(x)]_\beta$, $[U]_\beta^\gamma[h_0(x)]_\beta$ and $[U(h_0(x))]_\gamma$.